

How *Not* to Construct an Asymptotically de Sitter Universe

by

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ABSTRACT

Observational evidence suggests that our universe is currently evolving towards an asymptotically de Sitter future. Unfortunately and in spite of much recent attention, various quantum, holographic and cosmological aspects of de Sitter space remain quite enigmatic. With such intrigue in mind, this paper considers the “construction” of a toy model that describes an asymptotically de Sitter universe. More specifically, we add fluid-like matter to an otherwise purely de Sitter spacetime, formulate the relevant solutions and then discuss the cosmological and holographic implications. If the objective is to construct an asymptotically de Sitter universe that is free of singularities and has a straightforward holographic interpretation, then the results of this analysis are decidedly negative. Nonetheless, this toy model nicely illustrates the pitfalls that might be encountered in a more realistic type of construction.

1 Introduction

Recent astronomical observations have implied that the physical universe is currently in a phase of acceleration [1]. If this acceleration happens to be eternal, then our universe must eventually exhibit a cosmological event horizon. Furthermore, the universe will likely face an asymptotically de Sitter future; that is, a future spacetime that is dominated by a fixed, positive cosmological constant. It should be pointed out, however, that alternative scenarios may still account for this accelerating phase. For instance, rather than a cosmological constant *per se*, a dilatonic scalar field rolling in a stable potential towards a vanishing minimum; also known as “quintessence” [2, 3]. Nevertheless, some recent studies have demonstrated [4, 5] that the problematic features of an asymptotically de Sitter future, as discussed later in the section, do indeed persist in these alternative cosmologies.

The empirical evidence of an accelerating universe, as well as general curiosity, has provoked a recent flurry of research activity in the realm of de Sitter (dS) space and, more generally, asymptotically de Sitter (AsdS) spacetimes. In particular, holography [6, 7] in a de Sitter setting has sparked significant interest. (See Ref.[8] for a list of relevant citations, as well as Refs.[9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25] for particularly recent work.) However, in spite of all this attention, many aspects of AsdS spacetimes remain quite enigmatic. As a consequence, it has often been conceptually difficult to interpret (i) string theory, (ii) holography and/or (iii) physical observables in a de Sitter-cosmological framework. Next, let us briefly discuss some of the more perplexing issues; with the discussion categorized, perhaps arbitrarily, in terms of the above list of topics.

(i) *String Theory*:¹ Although a definitive theory of quantum gravity is still lacking, the consensus of opinion suggests that string theory - and/or its co-conspirator, M-theory - will have a significant role to play in the ultimate, fundamental theory. Unfortunately, string theory, as we currently understand it, does not appear to be compatible with de Sitter space. To explain this apparent impasse, let us consider the following. String theory is, of course, an inherently supersymmetric construction; thus implying a vanishing

¹For a review on string theory and M-theory that is aimed at the “lay-physicist”, see Ref.[26]. Note that, in the current discussion, our usage of string theory typically implies M-theory as well.

cosmological constant, as long as supersymmetry (SUSY) is preserved. This observation is, in itself, not problematic, considering that SUSY is clearly broken in our physical universe. So the challenge is really to find a SUSY-breaking string-theoretical vacuum that leaves us with an approximately flat, positively curved spacetime (as dictated by observational evidence).

Alas, any attempt in the prescribed direction has failed to provide an acceptable description of our physical reality. Firstly, in a context of perturbative or weakly coupled string theory, SUSY-violating vacua tend to have tachyonic instabilities. These, in turn, either give rise to a large, negative vacuum energy or drive the theory into an unphysical regime of decoupled gravity [27]. Secondly, in a context of non-perturbative string theory or matrix theory [28], any relevant vacuum collapses into a singular spacetime when SUSY is broken. Technically speaking, this collapse can be viewed as a consequence of a pertinent “no-go” theorem [29]. In a more fundamental sense, this failure can probably be attributed to de Sitter space having a finite number of degrees of freedom (see below); in complete contrast to matrix theory’s implication of an infinitely large Hilbert space.

(ii) *Holography*: The holographic principle, in its most basic sense, places an upper limit on the amount of information (i.e., entropy) that can be stored in a given region of spacetime [6, 7]. The generality of this statement can readily be seen by way of a covariant entropy bound that was first proposed by Bousso [30]. In particular, this bound directly relates the area of a given spatial surface to the maximal entropy passing through appropriately generated light sheets. Applying this formalism, Bousso went on to demonstrate that the entropy of pure de Sitter space will serve as an upper bound on the “accessible” entropy² in any AsdS spacetime [31].

As previously indicated, this finite entropic bound implies an incompatibility between string (or matrix) theory and de Sitter space. The same basic difficulty arises in the (conjectured) holographic duality between an $n+1$ -dimensional de Sitter bulk and an n -dimensional conformal field theory (CFT) [32].³ That is to say, the holographic dual of an AsdS spacetime - a Euclidean CFT that “lives” on a spacelike asymptotic boundary - is described by an infinite-dimensional Hilbert space and (therefore) has access to

²Accessible versus inaccessible entropy essentially makes the distinction between those degrees of freedom that can or cannot influence (and be influenced by) a given experiment.

³If valid, such a dS/CFT duality follows, by way of analogy, from the highly successful anti-de Sitter/CFT correspondence [33, 34, 35].

an infinite number of degrees of freedom. Equally distressing, the entropic upper bound seems to imply (at least indirectly via a closely related mass bound [36]) that the dual boundary theory is a non-unitary one [37, 38, 8].

While considering the paradoxical implications of the entropy bound, one should keep in mind that the holographic principle does *not*, on its own accord, prescribe the existence of a dually related boundary theory [39].⁴ But, on the other hand, the evidence in favor of a de Sitter-based duality does indeed appear to be mounting (see the previously cited literature). However, for a very recent argument in opposition to a viable dS/CFT correspondence, see Ref.[22].

(iii) *Physical Observables*: The feasibility of any proposed cosmology depends on the existence of well-defined observables that intelligent “passengers” can make sense of. For sake of argument, let us consider a (very) hypothetical asymptotically flat universe. In this case, the physical observables can be defined (and only defined) in terms of the S-matrix elements of asymptotically free particle states. An asymptotically anti-de Sitter universe provides a similar picture, except that the S-matrix is replaced by the boundary correlators of bulk fields [33, 34, 35]. What do both of these hypothetical cosmologies have in common? Precisely that there is an infinitely large asymptotic boundary (at spatial infinity) where the bulk degrees of freedom can separate into free particles.

With the above discussion in mind, we can now see that observables are, at best, poorly defined in an AsdS cosmology (and, in fact, any cosmology that admits a causal event horizon [2, 3]). First of all, there is no spatial infinity to speak of. There is, at least, a future infinity; however, for any given observer, this is nothing but a singular point. That is, an observer’s causal future shrinks to nothingness in the asymptotic limit. To put it another way, any two events that are sufficiently close to temporal infinity will have no common future and (therefore) immeasurable correlations. Consequently, any related S-matrix-like construction should have no physical meaning.

With regard to the above conundrum, Witten has suggested [40] that one can still construct a sort of “S-vector”; the elements of which measure the probability of a given initial state (at past infinity) to evolve into a given final state (at future infinity). This would allow the calculation of quantities

⁴In this regard, holography can be viewed as a necessary but *insufficient* prerequisite for establishing the anti-de Sitter/CFT correspondence.

that he refers to as “meta-observables”. However, it is not entirely clear as to what type of observer could make sense of meta-observables, insofar as it would require a global perspective that is outside the realm of “common” passengers. In another interesting, related discussion, Banks and Fischler [41] have proposed an S-like matrix that interpolates between a unique initial state (presumably, a “big bang”) and the states on an observer’s cosmological horizon.

It would seem safe to assume that the study of AsdS cosmologies has now been sufficiently motivated, and we proceed in this spirit. Before discussing the actual content of the current program, let us first consider a recent, topical paper by Leblond *et al* [18]. This informative work covered many aspects of de Sitter space; one of which was the “construction” of an AsdS spacetime containing non-trivial bulk fields. In particular, this “matter” was formulated in terms of a dilatonic scalar, with the gravity-dilaton coupling being described by a rather exotic potential.⁵ One might wonder if a similar construction can be achieved in terms of more “conventional” matter. This, in a nutshell, is the premise that underlies the current treatment.

To keep things simple (and thus calculable), we limit considerations to a toy model for a “perturbed” de Sitter cosmology. More specifically, fluid-like matter is added to an otherwise purely de Sitter spacetime (of arbitrary dimensionality). We also constrain the matter to have a fixed equation of state and to satisfy the “causal energy condition” [42]. We do, however, consider matter with both a positive and negative energy density. The latter, somewhat unorthodox choice will be rationalized at an appropriate juncture.

Let us point out that the methodology employed here closely follows that of McInnes [9]. In fact, this author considered the specific case of negative-energy matter (in a 4-dimensional AsdS spacetime); the purpose being to investigate the feasibility of so-called “phantom” cosmologies [43]. Hence, with regard to the negative-energy case, there is some overlap between the two studies. Nevertheless, we feel that our perspective and interpretations are somewhat different.

The remainder of the paper is organized as follows. In Section 2, we

⁵Actually, the authors of Ref.[18] considered a pair of related potentials: a discontinuous (“step”) potential and its smooth-functional “analogue”. The latter form is more realistic but necessitated a numerical analysis.

introduce the pertinent formalism and establish the appropriate boundary conditions; enabling us to precisely describe the cosmological model of interest (see above). Early on, a distinction is made between a pair of very different cosmological scenarios: an AsdS spacetime perturbed with (A) positive-energy matter and (B) negative-energy matter. In Section 3, we explicitly formulate the spacetime metric for both Case A and B. After which, the results are interpreted and various implications are discussed. In Section 4, we reconsider the prior outcomes from a holographic point of view. In particular, we employ a standard prescription [44, 36] to calculate generalized C -functions and then study the associated renormalization group flows (for instance, [45, 46, 47, 48, 44, 36, 18]). Finally, Section 5 contains a summary and concluding remarks.

To cut to the chase, we find both cosmological models of interest to be problematic. The positive-energy case contains a true (curvature) singularity, whereas the negative-energy case may have irreconcilable difficulties from a holographic perspective. (This latter “breakdown” was first pointed out in Ref.[9]; however, unlike the prior work, we argue that this result may be open to interpretation.) Let us, therefore, emphasize that our current intention is *not* to construct a viable AsdS cosmology. Rather, we hope to illustrate, by way of a toy model, the pitfalls that may be encountered when and if a more realistic construction is attempted.

2 Preliminaries

We begin the formal analysis by considering an $n+1$ -dimensional Friedmann-Robertson-Walker (FRW) spacetime with a scale factor of $a(t)$ (where t is the cosmological time). The corresponding FRW metric can be expressed by way of the following line element:

$$ds^2 = -dt^2 + a^2(t)d\Sigma_{n,k}^2. \quad (1)$$

Here, $d\Sigma_{n,k}^2$ describes the n -dimensional (Euclidean) metric of a constant-time hypersurface, such that k parametrizes the curvature. For spherical, flat or hyperbolic spatial sections, k respectively takes on a value of +1, 0 or -1.

Let us assume, for sake of simplicity, that the spacetime is filled with a

perfect fluid of energy density ρ and pressure p .⁶ For matter such as this, it is known that Einstein's field equations ($G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}$)⁷ take on a Friedmann-like form (for instance, [49]). More specifically, one finds that the Einstein tensor equation reduces to the following pair of expressions:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{16\pi G}{n(n-1)}\rho, \quad (2)$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{n(n-1)}[(n-2)\rho + np], \quad (3)$$

where G is the $n+1$ -dimensional Newton gravitational constant and a dot indicates differentiation with respect to t .

After some manipulation, the above field equations can be utilized to derive the following energy-conservation relation:

$$\begin{aligned} \dot{\rho} &= -n\frac{\dot{a}}{a}[\rho + p] \\ &= -n\frac{\dot{a}}{a}[1 + \omega]\rho. \end{aligned} \quad (4)$$

In the lower line, we have introduced the usual equation-of-state parameter, ω , such that $p = \omega\rho$. Some well-known (fixed) choices for ω include a universe that is purely filled with radiative matter ($\omega = 1/n$), dust-like matter ($\omega = 0$), a positive cosmological constant ($\omega = -1$ with $\rho > 0$), and a negative cosmological constant ($\omega = -1$ with $\rho < 0$). Although, generally speaking, one could always regard ω as a function of space and time. Note that $|\omega| \leq 1$ is often enforced, which translates to the causal energy condition [42]. However, for sake of generality, this will not necessarily be the case in the analysis to follow.

Since our current focus is on “perturbed” de Sitter spacetimes, it is instructive to first consider the above picture for pure de Sitter space. In this case of positive cosmological constant (i.e., $\Lambda > 0$) and no other matter, $\omega_{dS} = -1$ and Eq.(4) reminds us that the energy density is a positive constant as well. For future reference, we present the de Sitter line element in

⁶A perfect fluid insinuates that the stress-energy tensor is diagonalized according to $T_{\nu}^{\mu} = \text{diag}[-\rho, p, p, \dots, p]$.

⁷Note that the cosmological constant, if non-vanishing, is assumed to be absorbed into the stress-energy tensor.

both planar coordinates ($k = 0$) and global coordinates ($k = 1$) [50]:

$$ds_{dS}^2|_{k=0} = -dt^2 + e^{2t/L} \sum_{i=1}^n dx_i^2, \quad (5)$$

$$ds_{dS}^2|_{k=1} = -dt^2 + \cosh^2\left(\frac{t}{L}\right) L^2 \left[d\chi^2 + \sin^2(\chi) d\Sigma_{n-1,1}^2 \right], \quad (6)$$

where L is the curvature radius defined by $\Lambda = n(n-1)/2L^2$. Note that the planar coordinate system only describes half of the de Sitter manifold, but one can obtain the other half by substituting a minus sign into the exponential.

Sometimes, it is more instructive when de Sitter space is expressed in conformal coordinates. We do so here for the global ($k = 1$) case:

$$ds_{dS}^2|_{k=1} = \frac{L^2}{\sin^2(\eta)} \left[-d\eta^2 + d\chi^2 + \sin^2(\chi) d\Sigma_{n-1,1}^2 \right], \quad (7)$$

where $d\eta = dt/L \cosh(t/L)$. With a suitable choice of integration constant, this translates into $\eta(t) = 2 \tan^{-1} [e^{t/L}]$. Take note of the finite extent of the conformal (i.e., Penrose) diagram. It has a “height” of $\Delta\eta = \eta(+\infty) - \eta(-\infty) = \pi$ and a “width” of $\Delta\chi = \pi - 0 = \pi$. Also of interest, any observer must suffer the indignities of both a past and future cosmological horizon. For example, an observer at $\chi = 0$ has a future horizon at $\chi = \pi - \eta$ and a past horizon at $\chi = \eta$. For further background on de Sitter space and plenty of fancy diagrams, one can consult Refs.[50, 18, 39].

An interesting feature of de Sitter space is that, regardless of the spatial slicing or choice of k , one finds that $a(t) \sim e^{t/L}$ as $|t| \rightarrow \infty$ [18]. We can use these observations, along with Eq.(2), to identify the (constant) energy density and pressure of a de Sitter cosmology:

$$\rho_{dS} = -p_{dS} = \frac{n(n-1)}{16\pi GL^2}. \quad (8)$$

Let us now concentrate on the scenario of interest; namely, incorporating matter (to be denoted by ϕ) into an otherwise purely de Sitter spacetime. The only stipulations on this supplementary matter will be that it is of a perfect-fluid form (so that Eqs.(2-4) remain in effect), it has a fixed equation of state (i.e., $\omega_\phi = p_\phi/\rho_\phi$ is a constant), and it obeys the causal entropy

bound (i.e., $|\omega_\phi| \leq 1$). The total energy and pressure are thus defined as follows:

$$\rho = \rho_{dS} + \rho_\phi, \quad (9)$$

$$p = p_{dS} + p_\phi = -\rho_{dS} + \omega_\phi \rho_\phi. \quad (10)$$

It should be kept in mind that the de Sitter contributions cancel off in the summation of ρ and p .

As implied by the above discussion, we will permit ρ_ϕ to be positive or negative; although the latter situation is in conflict with the (usually enforced) “weak energy condition” [42]. If some readers are bothered by such an exotic choice of matter, it may help to think of negative ρ_ϕ as describing the removal of energy (from pure de Sitter space) as opposed to the addition of negative energy. Indeed, a Schwarzschild-de Sitter black hole spacetime (for example) is really just a negative energy excitation of pure de Sitter space under static, spherically symmetric conditions [36].

The working hypothesis will be a universe that is asymptotically de Sitter (AsdS) in the distant past ($t = -\infty$) and, barring curvature singularities or divine intervention, evolves into an AsdS spacetime in the far future ($t = +\infty$). What will become quite evident, as the analysis proceeds, is two distinctive solution sets depending on the positivity/negativity of ρ_ϕ .⁸ To help clarify matters, we thus establish the following classifications:

- i) Case *A* for $\rho_\phi > 0$ (hence, $\rho + p > 0$);
- ii) Case *B* for $\rho_\phi < 0$ (hence, $\rho + p < 0$).

We take this opportune moment to (again) point out that a prior, related work by McInnes [9] studied the cosmology corresponding to Case *B* with $n = 3$ and $\omega_\phi < 0$.

Given our working hypothesis of an AsdS spacetime, it follows that we can write:

$$\rho|_{A,B} = \rho_{dS} \pm S(a) \quad (11)$$

where $S(a)$ is some well-behaved, positive function that satisfies $S(a) \rightarrow 0$ as a^2 (or $|t|$) $\rightarrow \infty$. Note that, where applicable, the left/right subscript label corresponds to the upper/lower sign. Also note that $S(a)$ is not *a priori* the same function in the two different cases.

⁸By way of Eq.(4), the causality bound, the AsdS initial condition and time-reversal symmetry, it can be shown that ρ_ϕ does not change its sign during the history of the universe.

At this point in the analysis, it proves convenient to define the following parameter:

$$\gamma \equiv n(1 + \omega_\phi). \quad (12)$$

Note that, strictly speaking, $0 \leq \gamma \leq 2n$. Although, if the resultant space-time is to be truly classified as a “perturbation” of de Sitter space, γ should really be limited to values much smaller than one. As it turns out, this distinction is never really an issue. (Contrary to this statement is, however, the last topic of Section 3.)

Given the above, Eq.(10) for the total pressure can now be re-expressed as follows:

$$p = -\frac{\gamma}{n}\rho_{dS} - \left[\frac{n - \gamma}{n} \right] \rho. \quad (13)$$

Further incorporating Eq.(11), we have:

$$p|_{A,B} = -\rho_{dS} \mp \left[\frac{n - \gamma}{n} \right] S(a) \quad (14)$$

or, more to the point:

$$\rho + p|_{A,B} = \pm \frac{\gamma}{n} S(a). \quad (15)$$

For Case *A* and *B* alike, the above outcomes can be substituted into the energy-conservation equation (4) to yield:

$$\dot{S} = -\gamma \frac{\dot{a}}{a} S(a). \quad (16)$$

Therefore:

$$\frac{dS}{da} = -\frac{\gamma}{a} S(a). \quad (17)$$

This differential equation can be trivially solved as follows:

$$S(a) = \alpha a^{-\gamma}, \quad (18)$$

where α is a positive constant. Note that the asymptotic behavior of $S(a)$ is precisely as anticipated.

3 Perturbed de Sitter Space

Our next objective will be to utilize the prior formalism (in particular, the most recent outcome and the field equations) in obtaining an explicit solution for the scale factor, $a(t)$. To substantially simplify this task, we will follow Ref.[9] and assume that the spatial sections are exactly flat (i.e., $k = 0$) and compact. Although this choice is motivated, in large part, by convenience, let us point out that (approximate) flatness follows from observational evidence, while compactness can be argued for by way of the following holographic consideration.

In regard to the argument for compactness, first recall (from Section 1) that there is a holographic upper limit on the number of *accessible* degrees of freedom in any AsdS spacetime [31]. Now consider that an arbitrary perturbation of de Sitter space will allow an entire Cauchy surface to be visible at some finite value of time [51, 18]. It is clear that, at least naively, these two statements are contradictory if the volume of the relevant Cauchy surface can increase without bound. Thus, compactness of the spatial slices naturally follows.

In any event, the intricacies of the spatial slicing will have no repercussions on the de Sitter picture at large values of $|t|$, where such information is “conveniently” concealed behind a cosmological horizon.⁹

For sake of definiteness (and roughly following [9]), let us regard the spatial sections as cubic tori that are parametrized by n angular coordinates, θ_i . That is, the FRW metric (1) now takes on the following form:

$$ds^2 = -dt^2 + a^2(t)\mathcal{A}^2 \sum_{i=1}^n d\theta_i^2, \quad (19)$$

where \mathcal{A} is some arbitrary length parameter.

We now return to the issue at hand - the evaluation of the scale factor - and reconsider the first-order field equation (2). Also incorporating Eqs.(8,11,18) and setting $k = 0$, we have:

$$\left(\frac{\dot{a}}{a}\right)_{A,B}^2 = \frac{1}{L^2} \pm \frac{16\pi G}{n(n-1)} \alpha a^{-\gamma}. \quad (20)$$

⁹Any future life-form trying to cope with the bleak final stages of an AsdS cosmology [40] would likely argue that this information loss is anything but convenient.

The above equation can trivially be solved, provided that α has been identified with $n(n-1)/16\pi GL^2$. Accepting this quite reasonable identification, we find (for Case *A* and *B*, respectively):

$$a(t)|_A = \sinh\left(\frac{2}{\gamma}\right) \left(\frac{\gamma(t-t_o)}{2L}\right), \quad (21)$$

$$a(t)|_B = \cosh\left(\frac{2}{\gamma}\right) \left(\frac{\gamma(t-t_o)}{2L}\right). \quad (22)$$

Note that t_o is a constant of integration, which, without loss of generality, will subsequently be set to vanish.

We can finally display the respective solutions with their time dependence explicitly indicated:

$$ds_A^2 = -dt^2 + \mathcal{A}^2 \sinh\left(\frac{4}{\gamma}\right) \left(\frac{\gamma t}{2L}\right) \sum_{i=1}^n d\theta_i^2, \quad (23)$$

$$ds_B^2 = -dt^2 + \mathcal{A}^2 \cosh\left(\frac{4}{\gamma}\right) \left(\frac{\gamma t}{2L}\right) \sum_{i=1}^n d\theta_i^2. \quad (24)$$

As a check on consistency, let us examine the asymptotic behavior of these solutions. Setting $x_i = 2^{-2/\gamma} \mathcal{A} \theta_i$ and taking $|t| \gg 1$, we obtain the following expression in both cases:

$$ds^2 \approx -dt^2 + e^{2|t|/L} \sum_{i=1}^n dx_i^2. \quad (25)$$

Notably, this asymptotic form agrees with Eq.(5), which describes a purely de Sitter spacetime with flat spatial slicing. Moreover, this form is, as previously discussed, the unique FRW description of any de Sitter space in the limit of large $|t|$ [18]. Thus, our toy model displays the anticipated AsdS behavior, regardless of the sign of the energy density.

Next, we comment on some of the implications of these solutions, starting with Case *A*. It is quite clear that the corresponding scale factor (21) has a zero on the constant-time surface defined by $t = 0$. What is not yet so evident is the physical significance of this zero. That is to say, does it represent: (i) a singularity in the spacetime beyond which time can not flow or (ii) a Cauchy null surface beyond which the solution can (and should)

be analytically continued? The simplest way to resolve this dilemma is to consider the scalar curvature on the surface in question. Calculating the prescribed quantity, we find:

$$R|_A = \frac{n(n+1)}{L^2} + \frac{n}{L^2} [n+1-\gamma] \frac{1}{\sinh^2\left(\frac{\gamma t}{2L}\right)} \quad (26)$$

and, for completeness:

$$R|_B = \frac{n(n+1)}{L^2} - \frac{n}{L^2} [n+1-\gamma] \frac{1}{\cosh^2\left(\frac{\gamma t}{2L}\right)}. \quad (27)$$

We see that, for Case *A*, the scalar curvature (26) “blows up” at $t = 0$. Hence, the corresponding zero in the metric does indeed represent a true singularity in the fabric of spacetime. Physically, this outcome can be perceived as a “big crunch” at which the universe collapses into a singular point.¹⁰ Moreover, we can, at least conjecturally, view this outcome as a manifestation of the AsdS entropy bound [31]. To re-clarify, it has been demonstrated that the entropy of pure de Sitter space ($\sim L^n/G$) should serve as an upper bound on the number of degrees of freedom that can experimentally be probed in an AsdS spacetime.¹¹ With the addition of positive-energy matter to an otherwise purely de Sitter spacetime, it follows, at least naively, that this entropic upper bound has somewhere been violated. Consequently, any chance of a final AsdS phase has effectively been thwarted.

With regard to the singular nature of Case *A*, let us further point out the following. Since this cosmological scenario describes the addition of (positive-energy) matter, rather than entropy *per se*, it is (perhaps) most appropriate to interpret the singularity as a violation of a proposed AsdS mass bound [36].¹² To elaborate, it has recently been argued that the mass of pure de

¹⁰Alternatively, by way of time-reversal symmetry (as implied in this toy model), one can just as easily regard this singularity as a “big bang”. The choice of bang versus crunch depends on whether the singularity precedes or proceeds time evolution. Until one invokes the second law of thermodynamics to point the way, it is really just a matter of taste.

¹¹In qualifying the degrees of freedom, we are making the distinction between those that are accessible versus inaccessible. Accessible degrees of freedom can both influence and be influenced by a given experiment. That is, the experiment in question determines a “causal diamond” or accessible domain of the spacetime [31].

¹²On the other hand, the entropy [31] and mass [36] bounds are closely related and may actually represent different semantics for the same physical principle.

Sitter space should serve as an upper bound on the total mass of any AsdS spacetime. Moreover, the authors of Ref.[36] have stressed that a violation of this bound should induce a cosmological singularity. Notably, this last point is clearly supported by our findings.

Let us now consider Case *B*, for which the plot changes dramatically. For this scenario, the scale factor (22) and scalar curvature (27) are both clearly regular throughout the spacetime manifold. Therefore, a perturbed de Sitter universe that remains AsdS, in both the distant past and the far future, has truly been realized. This outcome is not particularly surprising for two reasons. First of all, technically speaking, the AsdS entropy bound [31] only has validity when the null energy condition (which forbids negative-energy matter [42]) has been satisfied. Secondly, on a more intuitive level, this negative-energy case can effectively be viewed as a removal of positive-energy matter from an otherwise purely de Sitter spacetime. That is, Case *B* translates into, if anything, a reduction in the number of accessible degrees of freedom. Thus, from the viewpoint of either relevant bound (mass [36] or entropy [31]), there is nothing to obstruct this cosmology from persevering into the far future.

As a topical aside, we point out that Case *B* provides a concrete realization of the so-called “tall” and “wide” universes that have recently been advocated by Leblond *et al* [18]. To make a tall-story short, Gao and Wald [51] have demonstrated that a (reasonably) generic perturbation of de Sitter space leads to an increase in the height of its conformal diagram (see the discussion following Eq.(7)). This suggests that, given a perturbed but singularity-free de Sitter spacetime, a complete Cauchy surface will become entirely visible to an appropriate observer at a finite value of time. One can further envision an AsdS universe that is not only tall (in the above sense) but also arbitrarily wide; that is, the spatial slices and (therefore) visible Cauchy surface could well have an arbitrarily large volume.

How does the above discussion apply to the spacetime described by Case *B*? As it so happens, one can make the full extent of conformal time arbitrarily large by taking γ to be arbitrarily small [9]. (Recall that $0 \leq \gamma \leq 2n$ is the only formal stipulation and we ethically prefer a small value of γ in any case.) To substantiate this claim, let us first rewrite Eq.(24) in terms of

conformal coordinates:

$$ds_B^2 = L^2 \cosh\left(\frac{4}{\gamma}\right) \left(\frac{\gamma t}{2L}\right) \left[-d\eta^2 + \left(\frac{\mathcal{A}}{L}\right)^2 \sum_{i=1}^n d\theta_i^2\right]. \quad (28)$$

The full extent of conformal time is then given by:

$$\Delta\eta = \frac{1}{L} \int_{-\infty}^{+\infty} \frac{dt}{\cosh\left(\frac{2}{\gamma}\right) \left(\frac{\gamma t}{2L}\right)} = \frac{2}{\gamma} \int_0^\pi \sin\left(\frac{2}{\gamma}-1\right)(\xi) d\xi. \quad (29)$$

Even without an explicit evaluation of this integral, it is sufficiently evident that $\Delta\eta$ can be made arbitrarily large in the manner prescribed above. Furthermore, the spatial sections, although described as compact, can be made to have an arbitrarily large (but still finite) volume with a suitably large choice for \mathcal{A} . Thus, we can arrange the Case- B universe to be both tall and wide enough to fulfill anyone's expectations, no matter how greedy. What makes this outcome somewhat of a puzzle is that the analysis of Gao and Wald [51], which conceptually presupposes the tall/wide construction [18], only applies to perturbations satisfying the null energy condition. Case B is, by hypothesis, clearly in violation of this condition, and we presently have no explanation for this phenomenon.

4 RG Flows

As it is well known, there are many indicators that AsdS spacetimes can be holographically described by an appropriate conformal field theory (CFT). In particular, this dually related CFT is a Euclidean one that lives on a spacelike asymptotic boundary of the bulk spacetime [32]. (Consult Section 1 for further references.) If valid, such a dS/CFT duality follows, by way of analogy, from the celebrated anti-de Sitter (AdS)/CFT correspondence [33, 34, 35]. It remains unclear, however, as to what extent anti-de Sitter holographic features can be generically extrapolated into the de Sitter-based duality.¹³ Nonetheless, many such features do indeed persevere into a de Sitter realm; including the intriguing phenomena of renormalization group (RG) flows.

¹³For a recent critique on the dS/CFT correspondence, see Ref.[22].

Before proceeding to the analysis, let us discuss the premise behind holographic RG flows.¹⁴ Firstly, from an AdS/CFT perspective, it has been established that the monotonic evolution of a relevant bulk parameter induces a “flow” in the renormalization scale of the boundary theory and *vice versa*. (Note that this renormalization scale determines the ultraviolet cutoff or lattice spacing of the dual CFT.) This picture follows, in large part, from the ultraviolet/infrared (UV/IR) correspondence [52, 53]. That is, large distances (IR) in the anti-de Sitter bulk correspond to small distances (UV) on the conformal boundary and *vice versa*.

It should be kept in mind that a holographic RG flow depends on the existence of a generalized C -function. This follows, by way of analogy, with C -functions in a two-dimensional CFT context [54]. For AdS/CFT holographies in particular, the generalized C -function should, if properly identified, exhibit various monotonicity properties that are reflective of the underlying UV/IR duality [52, 53].

Next, let us consider RG flows in a dS/CFT holographic context. As observed by Strominger [44] (also see [36]), time evolution in a purely de Sitter bulk will generate conformal-symmetry transformations on its asymptotic boundaries (spacelike past infinity and future infinity). Particularly significant to prospective RG flows, this conformal symmetry will be jeopardized when the bulk spacetime is “only” an AsdS one. In this regard, it was argued [44] that AsdS time evolution will naturally induce a RG flow between a pair of conformal fixed points. These fixed points occur in the asymptotic past and future, where the symmetries of pure de Sitter space ultimately emerge. Moreover, Strominger proposed an associated C -function of the form [44]:¹⁵

$$C_{dim=n+1} \sim \left| \left(\frac{\dot{a}(t)}{a(t)} \right)^{-(n-1)} \right|. \quad (30)$$

For an AsdS spacetime with flat spatial slicing (and presuming the weak energy condition [42]), it has been shown that this definition of C displays appropriate monotonicity properties [44, 36].¹⁶ In particular, as time evolves

¹⁴See, for instance, Refs.[45, 46, 47, 48, 44, 36, 18] for further, pertinent discussion.

¹⁵This formulation is identical to the conventional AdS/CFT C -function [45, 46, 47]; except that the anti-de Sitter scale factor is a function of radial distance rather than time.

¹⁶Recently, Leblond *et al* have proposed a more generalized version of this C -function that can be applied to any choice of spatial slicing [18]. For our present considerations, however, Eq.(30) is sufficient.

forward, C flows to the UV (i.e., increases) in an expanding universe and flows to the IR in a contracting universe. Note that such monotonic behavior is often represented as being a “ C -theorem”. (Further note that, for a universe with both a contracting and expanding phase, one must abandon the conventional wisdom of C flowing in a strict monotonic fashion. For further discussion on this caveat, see Ref.[18].)

Let us now incorporate the above concepts into the framework of the current study. We begin here by considering positive-energy perturbations of pure de Sitter space; that is, Case A . Recalling Eq.(21) for the relevant scale factor, we obtain the following C -function (30) (up to irrelevant constant factors):

$$C_A \sim \left| \tanh^{(n-1)} \left(\frac{\gamma t}{2L} \right) \right|. \quad (31)$$

To ascertain the behavior of C_A as time evolves, let us consider the following derivative (again neglecting constant factors):

$$\frac{1}{C_A} \frac{\partial C_A}{\partial t} \sim \frac{t}{|t|} \frac{1}{\cosh^2 \left(\frac{\gamma t}{2L} \right)}. \quad (32)$$

Upon inspection of this result, one can see that C_A behaves precisely as anticipated. To be more specific, if $-\infty < t < 0$ (i.e., a contracting universe), $\dot{C}_A < 0$ and C_A flows monotonically to the IR. Meanwhile, if $0 < t < +\infty$ (i.e., an expanding universe), $\dot{C}_A > 0$ and C_A flows monotonically to the UV. As a point of interest, let us recall that, because of a curvature singularity, Case A can describe a universe that is contracting or expanding but incapable of doing both. Hence, for any specific model, C_A experiences a strictly monotonic flow for all relevant time.

Let us now reconsider Case B , describing negative-energy perturbations of de Sitter space.¹⁷ In this case, the appropriate scale factor is given by Eq.(22), which leads to the following results:

$$C_B \sim \left| \coth^{(n-1)} \left(\frac{\gamma t}{2L} \right) \right|, \quad (33)$$

$$\frac{1}{C_B} \frac{\partial C_B}{\partial t} \sim -\frac{t}{|t|} \frac{1}{\sinh^2 \left(\frac{\gamma t}{2L} \right)}. \quad (34)$$

¹⁷Note that a similar analysis has been carried out by McInnes in Ref.[9].

Here, the story deviates radically from what we have found above. For instance, during the contracting phase ($t < 0$), $\dot{C}_B > 0$ and C_B flows monotonically to the UV. Meanwhile, during the expanding phase ($t > 0$), $\dot{C}_B < 0$ and C_B flows monotonically to the IR. This behavior is diametrically opposed to the expectations of the conventional bulk time/RG flow duality. The question being: can we provide a physical picture that somehow resolves this conundrum?

Firstly, let us take the holographic duality literally and presume that the forward evolution of time constitutes a flow towards the UV/IR when the universe is expanding/contracting. With this strict interpretation, the only feasible circumvention would be to assume that the universe remains permanently fixed at $t = 0$; that is, “time stands still”. On behalf of this viewpoint, we point out that the time derivative of the scale factor vanishes at $t = 0$; meaning that this surface represents a fixed point (infinite UV) in the holographic dual. With regard to this line of reasoning, it is interesting to recall our prior analogy between the Case- B spacetime and a Schwarzschild-de Sitter black hole. To reiterate, both of these solutions can be identified as AsdS spacetimes having negative energy relative to pure de Sitter space. If we extrapolate this identification to the extreme and presume that the two solutions are essentially just different pictures of the same physical situation, then it follows that Case B should have a perfectly valid description in static coordinates (namely, the $t = 0$ spatial slice). Note that the compactness of the $t = 0$ spatial section, as implied by earlier analysis, is not of issue in this analogy; we can make the volume of this section arbitrarily large by increasing the “width” parameter, \mathcal{A} .

On the other hand, it is not quite clear, at least in this exotic case, how literally one should take the definition of forward time as dictated by the implied holographic duality. In fact, the relevant “ C -theorem” [44, 36, 18] assumes that any additional matter satisfies $\rho > 0$ and $\rho > |p|$ (i.e., the weak energy condition). By hypothesis, Case B fails the first of these conditions, and so we could have anticipated the unorthodox outcome *a priori*. Perhaps, there should be an analogous C -theorem for negative-energy scenarios whereby the flow directions are reversed. To put it another way, what exactly constitutes the direction of time flow, in any model exhibiting time-reversal symmetry, is not particularly clear.

5 Conclusion

In summary, we have been studying a toy cosmological model that describes a “perturbed” de Sitter spacetime. More specifically, we have considered some of the implications (both cosmological and holographic) when matter is incorporated into an otherwise purely de Sitter spacetime of arbitrary dimensionality. This “supplementary” matter was assumed to have a perfect-fluid description, have a fixed equation of state, and satisfy the causal energy condition. On a more generic note, we allowed the matter to have either a positive (Case *A*) or negative (Case *B*) energy density. Although the latter case describes exotic matter in violation of the null energy condition, we noted the following alternative interpretation: the “subtraction” of positive-energy matter from an otherwise purely de Sitter spacetime.

After some preliminary background and analysis, we were able to derive an analytical expression for the metric in both of the pertinent cases. (Here, we employed a methodology that was first used by McInnes in a related work [9].) To obtain these precise formulations, it was necessary to establish some boundary conditions on our FRW cosmological framework. In this regard, we proposed a universe that is asymptotically de Sitter in the distant past and, barring singularities, AsdS in the far future. We also stipulated that the constant-time spatial slices are compact and flat. Although this choice of slicing was motivated, in large part, by convenience, we also provided arguments on the basis of observational and holographic considerations. Ultimately, we were able to show that both solutions exhibited the correct asymptotic behavior.

With the solutions of interest explicitly realized, we went on to consider the cosmological implications of these outcomes. First of all, the positive-energy case (*A*) was shown to have an unavoidable singularity at $t = 0$. We identified this singularity as being a “big crunch” (or a “big bang” in the time-reversed scenario). We then argued that such singular behavior can be viewed as a direct manifestation of violating certain holographic bounds. It is significant that these bounds place an upper limit on the accessible entropy and total mass of any spacetime with a positive cosmological constant [31, 36]. Conversely, the negative-energy case (*B*) turned out to be regular throughout the spacetime manifold. This was not unexpected, inasmuch as matter of this nature does not jeopardize the holographic bounds. A perhaps less obvious outcome is that such a spacetime can provide a concrete realization

of an arbitrarily “tall” and “wide” AsdS-universe [18]. This was shown to be possible, in spite of Case *B* clearly violating the null energy condition and (therefore) nullifying a critical antecedent of the tall-universe construction [51].

The final phase of our analysis focused on holographic renormalization group flows. Such RG flows are expected to be induced on AsdS-spacetime boundaries as a consequence of the (conjectured) dS/CFT duality [32]. For both cases of interest, we employed a standard prescription [44] to calculate a generalized *C*-function. Moreover, we tested the resultant functions to determine if they describe RG flows that conform to the relevant *C*-theorem [36, 18]. It was ultimately demonstrated that the positive-energy case (*A*) could indeed satisfy the pertinent theorem. On the other hand, time evolution in Case *B* induced a RG flow in the “wrong” direction. (On the plus side of the ledger, the Case-*B* flow was shown to be monotonic up to an anticipated “bounce” at $t = 0$.)

Although Case *B* essentially failed its holographic “litmus test”, we did express a couple of salvational viewpoints. Firstly, we conjectured that the Case-*B* solution - specifically, the $t = 0$ spatial slice - can effectively be viewed as an eternally static universe. The premise underlying this proposal is as follows. (i) The $t = 0$ surface describes an infinitely ultraviolet fixed point along the RG trajectory. (ii) There could well be a strong link between this space-time and a (static) Schwarzschild-de Sitter black hole, considering that both solutions represent a negative-energy excitation of pure de Sitter space [36]. Secondly, we suggested that the usual convention for defining time evolution in an AsdS universe (towards the UV/IR in an expanding/contracting phase) may not be applicable to this model or, in fact, any model that violates the weak energy condition. Certainly, the “arrow of time” seems a rather vague notion when the universe has an inherent time-reversal symmetry.

Regardless of one’s personal viewpoint, a better understanding of this “holographic failure” would seem to be in order. One possible avenue would be an investigation into how this perplexing behavior is encoded in the holographically dual theory. In this regard, it is worth noting that the so-called “Casimir entropy” of a dual CFT [55] can also be interpreted as a generalized *C*-function [56, 38]. Since the Casimir entropy is a direct property of a boundary theory, unlike our prior prescription for *C* (30), it is perhaps the more “fundamental” description of a given *C*-function. That is to say, the Casimir entropy may serve as a better indicator of how the time arrow should

be orientated. Unfortunately, it is not so clear how one goes about evaluating the Casimir entropy when the bulk theory is intrinsically non-static. One possibility may be to adapt a “mutual entropy” approach that has recently been advocated for the (total) CFT entropy [25].

In conclusion, the results of this paper establish, if nothing else, that it is no trivial task to construct an AsdS universe that is both regular throughout the manifold and holographically well understood. To achieve this, it would appear that significant anisotropy is required in the matter fields; which would, of course, be expected of any halfway realistic model. It would be interesting to extend the current treatment (i.e., the treatment of McInnes [9]) in this sort of direction, although numerical analysis would almost certainly be required. Finally, let us note that a “satisfactory” model (with regard to the above criteria) has indeed been formalized [18]. However, the “matter” was introduced via a dilatonic scalar along with a rather extravagant form of potential. It may still be of interest to construct a suitable AsdS model with matter of a more conventional nature.

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